

VALUE DISTRIBUTION THEORY OF q -DIFFERENCES IN SEVERAL COMPLEX VARIABLES

TINGBIN CAO AND RISTO KORHONEN

ABSTRACT. In this paper, q -difference analogues of several central results in value distribution theory of several complex variables are obtained. The main result is the q -difference second main theorem for hypersurfaces. In addition, q -difference versions of the logarithmic derivative lemma, the second main theorem for hyperplanes, Picard's theorem, and the Tumura-Clunie theorem, are included.

1. INTRODUCTION

The most striking result in the Nevanlinna theory of value distribution is the second main theorem, which is an inequality relating two leading quantities in the value distribution theory. One of these quantities is the characteristic function, which measures the rate of growth of a function or a map, and the other quantity is the counting function, which tells the size of the preimages of points or sets. Since 1925, when R. Nevanlinna [31] established the value distribution theory for meromorphic functions in the complex plane \mathbb{C} , many forms of the second main theorem for holomorphic maps, as well as meromorphic maps, on various contexts were found. For example, in 1933, H. Cartan [7] extended Nevanlinna's second main theorem for the case of holomorphic curves into complex projective spaces sharing hyperplanes in general position. Later, Nochka extended the Cartan's second main theorem for the case of hyperplanes in subgeneral position; in 1941 Ahlfors, following Weils' work, gave a geometric approach to obtain the second main theorem. In 2004, Ru [34] extended the Cartan's second main theorem for the case of hypersurfaces. For the background of the Nevanlinna theory, we refer to [20, 43, 33, 32].

Recently, in order to consider the growth of entire or meromorphic solutions of complex difference equations, the difference analogues of the second main theorem for meromorphic functions or maps were established. In 2006, Halburd and Korhonen [18] obtained a difference analogue of the second main theorem for meromorphic functions in the complex plane. Wong, Law and Wong [40], and Halburd, Korhonen and Tohge [19] obtained, independently, a difference analogue of the second main theorem for holomorphic curves into complex projective spaces intersecting with

2010 *Mathematics Subject Classification.* Primary 32H30; Secondary 30D35.

Key words and phrases. Meromorphic mapping; Second main theorem; q -difference; q -Casorati determinant; Hypersurface; Logarithmic derivative lemma; Tumura-Clunie theorem.

The first author was partially supported by the Natural Science Foundation of China (#11461042) and the Natural Science Foundation of Jiangxi Province in China (#20161BAB201007).

The second author was partially supported by the Academy of Finland grants (#286877) and (#268009).

hyperplanes in general position. Recently, Korhonen, Li and Tohge [25] considered the second main theorem for the case of slowly moving periodic hyperplanes.

The purpose of this paper is to investigate value distribution of q -differences of meromorphic functions and meromorphic mappings in several complex variables. The main result, Theorem 5.5 in Section 5, is so far the first q -difference second main theorem for hypersurfaces. In order to prove this results we, firstly, obtain a q -difference analogue of the logarithmic derivative lemma for meromorphic functions in several complex variables in Section 3, and a general q -difference version of the second main theorem for hyperplanes in Section 4. As an application of our main results, we get a q -difference Picard theorem in Section 6. At last in Section 7, we discuss the q -difference counterpart of the Tumura-Clunie theorem in several complex variables. Necessary notation has been given in the following section.

2. PRELIMINARIES

2.1. Set $\|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2}$ for $z = (z_1, \dots, z_m) \in \mathbb{C}^m$, for $r > 0$, define

$$B_m(r) := \{z \in \mathbb{C}^m : \|z\| \leq r\}, \quad S_m(r) := \{z \in \mathbb{C}^m : \|z\| = r\}.$$

Let $d = \partial + \bar{\partial}$, $d^c = (4\pi\sqrt{-1})^{-1}(\partial - \bar{\partial})$. Thus $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$. Write

$$\sigma_m(z) := (dd^c\|z\|^2)^{m-1}, \quad \eta_m(z) := d^c \log \|z\|^2 \wedge \sigma_m(z)$$

for $z \in \mathbb{C}^m \setminus \{0\}$.

For a divisor ν on \mathbb{C}^m we define the following counting function of ν by

$$N(r, \nu) = \int_1^r \frac{n(t)}{t^{2m-1}} dt \quad (1 < r < \infty),$$

where

$$n(t) = \begin{cases} \int_{|\nu| \cap B(t)} \nu(z) \sigma_m(z), & \text{if } m \geq 2; \\ \sum_{|z| \leq t} \nu(z), & \text{if } m = 1. \end{cases}$$

Let $\varphi (\not\equiv 0)$ be an entire function on \mathbb{C}^m . For $a \in \mathbb{C}^m$, we write $\varphi(z) = \sum_{i=0}^{\infty} P_i(z-a)$, where the term P_i is a homogeneous polynomial of degree i . We denote the zero-multiplicity of φ at a by $\nu_{\varphi}(a) = \min\{i : P_i \not\equiv 0\}$. Thus we can define a divisor ν_{φ} such that $\nu_{\varphi}(z)$ equals the zero multiplicity of φ at z in the sense of [11, Definition 2.1] whenever z is a regular point of an analytic set $|\nu_{\varphi}| := \overline{\{z \in \mathbb{C}^m : \nu_{\varphi}(z) \neq 0\}}$.

Letting h be a nonzero meromorphic function on \mathbb{C}^m with $h = \frac{h_0}{h_1}$ and $\dim(h_0^{-1}(0) \cap h_1^{-1}(0)) \leq m-2$, we define $\nu_h^0 := \nu_{h_0}, \nu_h^{\infty} := \nu_{h_1}$.

For a meromorphic function h on \mathbb{C}^m , we usually write $N(r, \frac{1}{h}) := N(r, \nu_h^0)$ and $N(r, h) := N(r, \nu_h^{\infty})$. The Jensen's Formular is given as

$$N(r, \frac{1}{h}) - N(r, h) = \int_{S_m(r)} \log |h| \eta_m(z) - \int_{S_m(1)} \log |h| \eta_m(z).$$

The Proximity function of h is defined by

$$m(r, h) = \int_{S_m(r)} \log^+ |h(z)| \eta_m(z),$$

where $\log^+ x := \max\{\log x, 0\}$ for any $x > 0$.

2.2. A meromorphic mapping $f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ is a holomorphic mapping from U into $\mathbb{P}^n(\mathbb{C})$, where U can be chosen so that $K_f \equiv \mathbb{C}^m \setminus U$ is an analytic subvariety of \mathbb{C}^m of codimension at least 2. Furthermore f can be represented by a holomorphic mapping of \mathbb{C}^m to \mathbb{C}^{n+1} such that

$$K_f = \{z \in \mathbb{C}^m : f_0(z) = \cdots = f_n(z) = 0\},$$

where f_0, \dots, f_n are holomorphic functions on \mathbb{C}^m . We say that $f = [f_0, \dots, f_n]$ is a reduced representation of f (the only factors common to f_0, \dots, f_n are units). If $g = hf$ for h any quotient of holomorphic functions on \mathbb{C}^m , then g will be called a representation of f (e.g. reduced if and only if h is holomorphic and a unit). Set $\|f\| = (\sum_{j=0}^n |f_j|^2)^{\frac{1}{2}}$. The Nevanlinna-Cartan's characteristic function for a meromorphic mapping f is defined by

$$\begin{aligned} T_f(r) &= \int_{S_m(r)} \log \|f\| \eta_m(z) - \int_{S_m(1)} \log \|f\| \eta_m(z) \\ &= \int_{S_m(r)} \log \max\{|f_0|, \dots, |f_n|\} \eta_m(z) + O(1) \quad (r > r_0 > 1). \end{aligned}$$

Note that $T_f(r)$ is independent of the choice of the reduced representation of f . The order of f is defined by

$$\zeta(f) := \limsup_{r \rightarrow \infty} \frac{\log^+ T_f(r)}{\log r}.$$

For $q \in \mathbb{C}^m \setminus \{0\}$, we denote by \mathcal{M} the set of all meromorphic functions over \mathbb{C}^m , by \mathcal{P}_q the set of all meromorphic functions $h \in \mathcal{M}$ satisfying $h(qz) \equiv h(z)$, and by \mathcal{P}_q^0 the set of all meromorphic functions in \mathcal{P}_q and having zero order. Obviously, then we have the inclusions $\mathcal{M} \supset \mathcal{P}_q \supset \mathcal{P}_q^0$.

We say that a meromorphic mapping f from \mathbb{C}^m into $\mathbb{P}^n(\mathbb{C})$ with a reduced representation $[f_0, \dots, f_n]$ is algebraically (linearly) nondegenerate over \mathcal{P}_q^0 if the entire functions f_0, \dots, f_n are algebraically (linearly) independent over \mathcal{P}_q^0 , and say that f is algebraically (linearly) nondegenerate over \mathbb{C} if the entire functions f_0, \dots, f_n are algebraically (linearly) independent over \mathbb{C} .

2.3. A hypersurface Q with degree d in $\mathbb{P}^n(\mathbb{C})$ is given by

$$Q = \{[x_0 : \cdots : x_n] : \sum_{I \in \mathcal{J}_d} a_I x^I = 0\},$$

where $\mathcal{J}_d = \{(i_0, \dots, i_n) \in \mathbb{N}_0^{n+1} : i_0 + \cdots + i_n = d\}$, $I = (i_0, \dots, i_n) \in \mathcal{J}_d$, $x^I = x_0^{i_0} \cdots x_n^{i_n}$ and $(x_0 : \cdots : x_n)$ are homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$. Denote by D the homogeneous polynomial associated with the hypersurface Q . If d is 1, then the hypersurface reduces to one hyperplane, denoted by H , as

$$H = \{[x_0 : \cdots : x_n] : a_0 x_0 + \cdots + a_n x_n = 0\}.$$

Set

$$Q(f(z)) := D \circ f(z) = \sum_{I \in \mathcal{J}_d} a_I f^I,$$

where $f^I = f_0^{i_0} \cdots f_n^{i_n}$. We recall the proximity function of f intersecting Q defined as

$$m_f(r, Q) = \int_{S_m(r)} \log \frac{\|f(z)\|^d \|a\|^d}{|Q(f(z))|} \eta_m(z),$$

where $\|a\| = (\sum_{I \in \mathcal{J}_d} |a_I|^2)^{1/2}$. Throughout this paper, we usually assume that $f(\mathbb{C}^m) \not\subset Q$ without a special statement. Then we have the first main theorem as follows:

$$m_f(r, Q) + N\left(r, \frac{1}{Q(f)}\right) = dT_f(r) + O(1).$$

Now let $\{Q_i\}_i^p$ be hypersurfaces of $\mathbb{P}^n(\mathbb{C})$. We say that the family of the hypersurfaces $\{Q_j\}_{j=1}^p$ are in general position in $\mathbb{P}^n(\mathbb{C})$ if for any subset $R \subset Q$ with the cardinality $\#R = n+1$, we have

$$\bigcap_{j \in R} Q_j = \emptyset.$$

That is, any $n+1$ homogeneous polynomials (forms) of $\{D_j(z)\}_{j=1}^p$ associated with the hypersurfaces $\{Q_j\}_{j=1}^p$ are linearly independent over \mathbb{C} .

2.4. Let f be a meromorphic mapping from \mathbb{C}^m into $\mathbb{P}^n(\mathbb{C})$. In what follows, $0 = (0, \dots, 0) \in \mathbb{C}^m$, $1 = (1, \dots, 1) \in \mathbb{C}^m$. For $q = (q_1, \dots, q_m)$ and $z = (z_1, \dots, z_m)$, we write $q + z = (q_1 + z_1, \dots, q_m + z_m)$, $qz = (q_1 z_1, \dots, q_m z_m)$. Denote the q -difference operator by

$$\Delta_q f := f(qz) - f(z) = f(q_1 z_1, q_2 z_2, \dots, q_m z_m) - f(z_1, z_2, \dots, z_m).$$

For $q \in \mathbb{C}^m \setminus \{0\}$, and a meromorphic mapping $f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = [f_0, \dots, f_n]$, we use the short notations

$$f(z) \equiv f := \overline{f}^{[0]}, f(qz) \equiv \overline{f} =: \overline{f}^{[1]}, f(q^2 z) \equiv \overline{\overline{f}} =: \overline{f}^{[2]}, \dots, f(q^k z) \equiv \overline{f}^{[k]}.$$

Then, analogously to the definitions of the Wronskian and the Casoratian determinants, the q -Casorati determinant of f is defined by

$$C(f) = C(f_0, \dots, f_n) = \begin{vmatrix} \frac{f_0}{f_0} & \frac{f_1}{f_1} & \cdots & \frac{f_n}{f_n} \\ \frac{f_0}{f_0} & \frac{f_1}{f_1} & \cdots & \frac{f_n}{f_n} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{f_0}^{[n]} & \overline{f_1}^{[n]} & \cdots & \overline{f_n}^{[n]} \end{vmatrix}_{(n+1) \times (n+1)}.$$

Given a real positive integer d , $\mathcal{J}_d = \{(i_0, \dots, i_n) \in \mathbb{N}_0^{n+1} : i_0 + \dots + i_n = d\}$. For any $I_j = (i_{j0}, \dots, i_{jn}) \in \mathcal{J}_d$ where $j \in \{1, \dots, M\}$, we set $f^{I_j} = f_0^{i_{j0}} \cdots f_n^{i_{jn}}$. Then q -Casorati determinant of f is given as

$$\tilde{C}(f) = C(f^{I_1}, \dots, f^{I_M}) = \begin{vmatrix} \frac{f^{I_1}}{f^{I_1}} & \frac{f^{I_2}}{f^{I_2}} & \cdots & \frac{f^{I_M}}{f^{I_M}} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{f^{I_1}}^{[M-1]} & \overline{f^{I_2}}^{[M-1]} & \cdots & \overline{f^{I_M}}^{[M-1]} \end{vmatrix}_{M \times M}.$$

Clearly, when $d = 1$ and $M = n+1$, we have $|\tilde{C}(f)| = |C(f)|$. Moreover, one can rearrange the order of I_1, \dots, I_M such that $\tilde{C}(f) = C(f)$ whenever $d = 1$ and $M = n+1$.

3. q -DIFFERENCE ANALOGUE OF THE LOGARITHMIC DERIVATIVE LEMMA IN SEVERAL COMPLEX VARIABLES

The original version of the logarithmic derivative lemma in one complex variable plays a key role in Nevanlinna theory for meromorphic functions in the complex plane, and is widely used in value distribution of meromorphic functions, differential equations in the complex plane, and so on. The first generalization of the logarithmic derivative lemma from one variable to several complex variables was given by Vitter [36], another proof was given by Biancofiore and Stoll [4].

In [2], a q -difference analogue of the logarithmic derivative lemma for meromorphic functions on the complex plane \mathbb{C} was obtained. Note that the assumption of f with zero order is sharp.

Theorem 3.1. [2, Theorem 1.1] *Let f be a non-constant zero-order meromorphic function on \mathbb{C} , and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = o(T(r, f))$$

on a set of logarithmic density 1.

It is natural to consider how to extend Theorem 3.1 to several complex variables. In [5], the first author of this paper tried to use the Biancofiore-Stoll method and found that there is one big technical difficulty in extending Theorem 3.1 to the case of several complex variables. Meanwhile, Wen [39] also attempted to use the same method and claimed that Theorem 3.1 is still true for a meromorphic function on \mathbb{C}^m and a given $q \in \mathbb{C}^m \setminus \{0\}$. Unfortunately, there is a gap in [39], which has to do with the same technical problem as the first author of this paper met in [5]. In fact, from the proof of [39, Lemma 5] one can see that the term $(\frac{R}{r})^{2n-2}$ is missing on the right hand side of the inequality in the statement of [39, Lemma 5], which means that the equality (2.3) in [39] cannot be obtained by taking $R = kr$ and making use of [39, Lemma 5 and Lemmas 6-8] in the proof of [39, Theorem 1]. This gap also effects the proofs of Theorems 9-13 in [39], as well as the proofs of the main results in [38], which all depend on [39, Theorem 1]. From the following result on the q -difference analogue of the logarithmic derivative lemma we conclude that all of these results remain valid under the additional assumption of $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$.

Here, we will adopt the method due to Stoll [35] and Fujimoto [14] to obtain a weak q -difference analogue of the logarithmic derivative lemma for a meromorphic function of several complex variables and a specially given $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$. This result generalizes Theorem 3.1, but it is still open whether the result remains true for any given $q \in \mathbb{C}^m \setminus \{0\}$.

Theorem 3.2. *Let f be a nonconstant zero-order meromorphic function on \mathbb{C}^m and let $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = \int_{S_m(r)} \log^+ \left| \frac{f(qz)}{f(z)} \right| \eta_m(z) = o(T_f(r))$$

for all $r = \|z\|$ on a set of logarithmic density one.

Proof. Let E_1 be the set of all points $\xi \in S_m(1)$ such that $\{z = u\xi : |u| < +\infty\} \subset K_f$ which is of measure zero in $S_m(1)$. For any $\xi \in S_m(1) \setminus E_1$, considering the

meromorphic function $f^\xi(u) := f(\xi u)$ of \mathbb{C}^1 , we have

$$T_{f^\xi}(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |f^\xi(re^{\sqrt{-1}\theta})| d\theta - \log |f(0)|,$$

and thus by [35, Lemmas 1.1–1.2] it follows (see also [14, pp. 33–34])

$$(1) \quad T_f(r) = \int_{S_m(1)} T_{f^\xi}(r) \eta_m(z).$$

Recall that the proximity function of the meromorphic function f on \mathbb{C}^m is defined by

$$m(r, f) = \int_{S_m(r)} \log^+ |f(z)| \eta_m(z).$$

For any $\xi \in S_m(1) \setminus E_1$, considering the meromorphic function $f^\xi(u) := f(\xi u)$ of \mathbb{C}^1 , we have

$$m(r, f^\xi) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f^\xi(re^{\sqrt{-1}\theta})| d\theta.$$

Then by [35, Lemmas 1.1–1.2], we also get

$$(2) \quad m(r, f) = \int_{S_m(1)} m(r, f^\xi) \eta_m(z).$$

Since $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$, $\tilde{q} \in \mathbb{C}^1 \setminus \{0\}$. For any $\xi \in S_m(1) \setminus E_1$, considering the meromorphic function $f^\xi(u) := f(\xi u)$ of \mathbb{C}^1 , we get from (2) that

$$\begin{aligned} m\left(r, \frac{f(qz)}{f(z)}\right) &= \int_{S_m(r)} \log^+ \left| \frac{f(qz)}{f(z)} \right| \eta_m(z) \\ &= \int_{S_m(1)} \left(\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f^\xi(|\tilde{q}u|e^{\sqrt{-1}\theta})}{f^\xi(|u|e^{\sqrt{-1}\theta})} \right| d\theta \right) \eta_m(z), \end{aligned}$$

where we denote $z = u\xi$ for any $\xi \in S_m(1)$. By [2, Lemma 5.1], we get that for all $r > 0$, $M > \max\{1, |\frac{v}{u}|\} = \|q\| = |\tilde{q}|$,

$$\begin{aligned} m\left(r, \frac{f^\xi(|\tilde{q}u|e^{\sqrt{-1}\theta})}{f^\xi(|u|e^{\sqrt{-1}\theta})}\right) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f^\xi(|\tilde{q}u|e^{\sqrt{-1}\theta})}{f^\xi(|u|e^{\sqrt{-1}\theta})} \right| d\theta \\ &\leq \frac{O(1)}{M} \left(T_{f^\xi}(Mr) + \log^+ \frac{1}{|f^\xi(0)|} \right), \end{aligned}$$

where $r = |u| = \|u\xi\| = \|z\|$. Therefore, together with (1), it follows from the two inequalities above that

$$\begin{aligned} m\left(r, \frac{f(qz)}{f(z)}\right) &= \int_{S_m(r)} \log^+ \left| \frac{f(qz)}{f(z)} \right| \eta_m(z) \\ &\leq \int_{S_m(1)} \left(\frac{O(1)}{M} \left(T_{f^\xi}(Mr) + \log^+ \frac{1}{|f^\xi(0)|} \right) \right) \eta_m(z) \\ &= \frac{O(1)}{M} \int_{S_m(1)} T_{f^\xi}(Mr) \eta_m(z) + O(1), \end{aligned}$$

namely,

$$(3) \quad m\left(r, \frac{f(qz)}{f(z)}\right) \leq \frac{O(1)}{M} T_f(Mr) + O(1).$$

The following is dealt similarly as in [2]. By choosing $M := 2^n$ and by applying [20, Lemma 4], we get $T_f(Mr) \leq 2T_f(r)$ on a set of logarithmic density one. Hence by [2, Lemma 5.3] we get from (3) that

$$m\left(r, \frac{f(qz)}{f(z)}\right) = o(T_f(r))$$

for all r on a set of logarithmic density one. \square

Since

$$\frac{f(z)}{f(qz)} = \frac{f[\frac{1}{q}(qz)]}{f(qz)}, \quad \frac{\bar{f}^{[k]}}{f(z)} = \frac{\bar{f}^{[k]}}{\bar{f}^{[k-1]}} \cdot \frac{\bar{f}^{[k-1]}}{\bar{f}^{[k-2]}} \cdots \frac{\bar{f}}{f(z)} \quad (k \in \mathbb{N}),$$

it follows immediately from Theorem 3.2 that

$$\int_{S_m(r)} \log^+ \left| \frac{f(z)}{f(qz)} \right| \eta_m(z) = o(T_f(r)),$$

$$\int_{S_m(r)} \log^+ \left| \frac{\bar{f}^{[k]}}{f(z)} \right| \eta_m(z) + \int_{S_m(r)} \log^+ \left| \frac{f(z)}{\bar{f}^{[k]}} \right| \eta_m(z) = o(T_f(r))$$

for all r on a set of logarithmic density one.

4. q -DIFFERENCE ANALOGUE OF THE SECOND MAIN THEOREM FOR HYPERPLANES

In 2007, the second main theorem for meromorphic functions of one variable was obtained by Barnett-Halburd-Korhonen-Morgan [2].

Theorem 4.1. [2] *Let f be a non-constant zero-order meromorphic function on \mathbb{C} , let $\Delta_q f \not\equiv 0$ and $q \in \mathbb{C} \setminus \{0\}$, and let $a_1, a_2, \dots, a_p \in \mathbb{C}$, $p \geq 2$, be distinct points. Then*

$$m(r, f) + \sum_{k=1}^p m\left(r, \frac{1}{f - a_k}\right) \leq 2T_f(r) - N_{\text{pair}}(r, f) + o(T_f(r))$$

on a set with a logarithmic density one, where

$$N_{\text{pair}}(r, f) := 2N(r, f) - N(r, \Delta_q f) + N\left(r, \frac{1}{\Delta_q f}\right).$$

In [5], Cao extended Theorem 4.1 to the case for meromorphic functions of several complex variables and a given number $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$, by directly applying the method of Stoll and Fujimoto (the standard process of averaging over the complex lines in \mathbb{C}^m) to the second main theorem of one variable, and obtained the following result.

Theorem 4.2. [5] *Let f be a nonconstant meromorphic function of order zero on \mathbb{C}^m such that $\Delta_q f \not\equiv 0$, where $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$. For any p distinct values $a_1, a_2, \dots, a_p \in \mathbb{P}(\mathbb{C})$,*

$$(p-2)T_f(r) \leq \sum_{j=1}^p N\left(r, \frac{1}{f - a_j}\right) - N\left(r, \frac{1}{C(f_0, f_1)}\right) + o(T_f(r))$$

on a set with a logarithmic density one, where $C(f_0, f_1)$ is the q -Casorati determinant of f with a reduced representation $f = [f_0, f_1]$.

It is known that holomorphic functions g_0, \dots, g_n on \mathbb{C}^m are linearly dependent over \mathbb{C}^m if and only if their Wronskian determinant $W(g_0, \dots, g_n)$ vanishes identically [12, Proposition 4.5]. By the definition of the characteristic function and using a similar discussion as in [15, Theorem 1.6, p. 47], one can easily get that for any meromorphic function h on \mathbb{C}^m and $q \in \mathbb{C}^m \setminus \{0\}$

$$T_{h(qz)}(r) = O\left(T_{h(z)}(|q|r)\right).$$

Then considering this fact and making use of almost the same discussion as in [19, Lemma 3.2], we can get the following Lemma 4.3 (ii). To prove (i) it is just not necessary to consider the growth of f in the proof of (ii). We omit the details.

Lemma 4.3. (i) Let $q \in \mathbb{C}^m \setminus \{0\}$. A meromorphic mapping $f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ with a reduced representation $[f_0, \dots, f_n]$ satisfies $C(f_0, \dots, f_n) \not\equiv 0$ if and only if f is linearly nondegenerate over the field \mathcal{P}_q .

(ii) Let $q \in \mathbb{C}^m \setminus \{0\}$. If a meromorphic mapping $f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ with a reduced representation $[f_0, \dots, f_n]$ satisfies $\zeta(f) = 0$, then $C(f_0, \dots, f_n) \not\equiv 0$ if and only if f is linearly nondegenerate over the field $\mathcal{P}_q^0(\subset \mathcal{P}_q)$.

The following result is a general version of the second main theorem for a meromorphic mapping f into a complex projective space intersecting hyperplanes, with the ramification term in terms of the q -Casorati determinant. Note that here we do not need a growth condition for f .

Theorem 4.4. Let $q \in \mathbb{C}^m \setminus \{0\}$ and $f = [f_0, \dots, f_n] : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping over the field \mathcal{P}_q . Let H_1, \dots, H_p (defining coefficient vectors a_1, \dots, a_p respectively) be hyperplanes located in general position in $\mathbb{P}^n(\mathbb{C})$. Let T be the set of all injective maps $\mu : \{0, 1, \dots, n\} \rightarrow \{1, \dots, p\}$ such that $a_{\mu(0)}, \dots, a_{\mu(n)}$ are linearly independent. Denote by $h_{\mu(l)} = \langle f(z), a_{\mu(l)} \rangle$ and $g_{\mu(l)} = \frac{h_{\mu(l)}}{h_{\mu(0)}}$ for all $0 \leq l \leq n$. Then we have

$$\begin{aligned} & \int_{S_m(r)} \max_K \sum_{k \in K} \lambda_{H_k}(f(z)) \eta_m(z) \\ & \leq \int_{S_m(r)} \max_{\mu \in T} \sum_{j=0}^n \log \frac{\|f(z)\| \|a_{\mu(j)}\|}{|\langle f(z), a_{\mu(j)} \rangle|} \eta_m(z) \\ & \leq (n+1)T_f(r) - N(r, \frac{1}{C(f_0, \dots, f_n)}) + O(1) \\ & \quad + \sum_{\mu \in T} \sum_{i_1 + \dots + i_n \leq \frac{n(n+1)}{2}} \sum_{l=1}^n \int_{S_m(r)} \log^+ \frac{|\bar{g}_{\mu(l)}^{[i_l]}|}{|g_{\mu(l)}|} \eta_m(z) \\ & \quad + \sum_{\mu \in T} \sum_{l=1}^n \int_{S_m(r)} \log^+ \frac{|g_{\mu(l)}|}{|\bar{g}_{\mu(l)}^{[l]}|} \eta_m(z) + \sum_{\mu \in T} \sum_{j=0}^n \left(N(r, \frac{1}{\bar{h}_{\mu(j)}^{[j]}}) - N(r, \frac{1}{h_{\mu(j)}}) \right), \end{aligned}$$

where $C(f_0, \dots, f_n)$ is the q -Casorati determinant of f , and the maximum is taken over all subsets K of $\{1, \dots, p\}$ such that a_j ($j \in K$) are linearly independent.

Proof. Let H_1, \dots, H_p be the given hyperplanes with coefficient vectors a_1, \dots, a_p in \mathbb{C}^{n+1} . Denote by $K \subset \{1, \dots, p\}$ such that a_k ($k \in K$) are linearly independent. Since $\{Q_j\}_{j=1}^p$ are located in general position in $\mathbb{P}^n(\mathbb{C})$, it follows that $\sharp K := k +$

$1 \leq n+1$. Let T be the set of all injective maps $\mu : \{0, 1, \dots, n\} \rightarrow \{1, \dots, p\}$ such that $a_{\mu(0)}, \dots, a_{\mu(n)}$ are linearly independent. Denote the Weil function by $\lambda_{H_j}(f(z)) = \log \frac{\|f(z)\| \|a_j\|}{|\langle f(z), a_j \rangle|}$ for all $j \in \{1, \dots, p\}$, where $\langle f(z), a_j \rangle$ means the inner product. Then we get

$$\begin{aligned}
& \int_{S_m(r)} \max_K \sum_{k \in K} \lambda_{H_k}(f(z)) \eta_m(z) \\
& \leq \int_{S_m(r)} \max_{\mu \in T} \sum_{j=0}^n \log \frac{\|f(z)\| \|a_{\mu(j)}\|}{|\langle f(z), a_{\mu(j)} \rangle|} \eta_m(z) \\
& \leq \int_{S_m(r)} \log \left\{ \max_{\mu \in T} \frac{\|f(z)\|^{n+1}}{|C(\langle f(z), a_{\mu(0)} \rangle, \dots, \langle f(z), a_{\mu(n)} \rangle)|} \right\} \eta_m(z) \\
& \quad + \int_{S_m(r)} \max_{\mu \in T} \left\{ \log \frac{|C(\langle f(z), a_{\mu(0)} \rangle, \dots, \langle f(z), a_{\mu(n)} \rangle)|}{\prod_{j=0}^n |\langle f(z), a_{\mu(j)} \rangle|} \right\} \eta_m(z) + O(1) \\
& \leq \int_{S_m(r)} \log \left\{ \sum_{\mu \in T} \frac{\|f(z)\|^{n+1}}{|C(\langle f(z), a_{\mu(0)} \rangle, \dots, \langle f(z), a_{\mu(n)} \rangle)|} \right\} \eta_m(z) \\
& \quad + \int_{S_m(r)} \sum_{\mu \in T} \log \left\{ \frac{|C(\langle f(z), a_{\mu(0)} \rangle, \dots, \langle f(z), a_{\mu(n)} \rangle)|}{\prod_{j=0}^n |\langle f(z), a_{\mu(j)} \rangle|} \right\} \eta_m(z) + O(1) \\
& := I_1 + I_2 + O(1),
\end{aligned}$$

where $C(\langle f(z), a_{\mu(0)} \rangle, \dots, \langle f(z), a_{\mu(n)} \rangle)$ denotes the q -Casorati determinant of $\langle f(z), a_{\mu(0)} \rangle, \dots, \langle f(z), a_{\mu(n)} \rangle$.

Set $h_{\mu(j)} := \langle f(z), a_{\mu(j)} \rangle$ and $a_{\mu(j)} = (a_{\mu(j),0}, \dots, a_{\mu(j),n})$ for all $0 \leq j \leq n$. Then we have

$$\begin{aligned}
& |C(\langle f(z), a_{\mu(0)} \rangle, \dots, \langle f(z), a_{\mu(n)} \rangle)| \\
& = \begin{vmatrix} h_{\mu(0)} & h_{\mu(1)} & \cdots & h_{\mu(n)} \\ \overline{h}_{\mu(0)}^{[1]} & \overline{h}_{\mu(1)}^{[1]} & \cdots & \overline{h}_{\mu(n)}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{h}_{\mu(0)}^{[n]} & \overline{h}_{\mu(1)}^{[n]} & \cdots & \overline{h}_{\mu(n)}^{[n]} \end{vmatrix} \\
& = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ \overline{f}_0^{[1]} & \overline{f}_1^{[1]} & \cdots & \overline{f}_n^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{f}_0^{[n]} & \overline{f}_n^{[n]} & \cdots & \overline{f}_n^{[n]} \end{vmatrix} \cdot \begin{vmatrix} a_{\mu(0),0} & a_{\mu(1),0} & \cdots & a_{\mu(n),0} \\ a_{\mu(0),1} & a_{\mu(1),1} & \cdots & a_{\mu(n),1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\mu(0),n} & a_{\mu(1),n} & \cdots & a_{\mu(n),n} \end{vmatrix} \\
& := |C(f_0, \dots, f_n)| \cdot A_{\mu}.
\end{aligned}$$

By Lemma 4.3 we know that $C(f_0, \dots, f_n) \neq 0$ since f is linearly nondegenerate over the field \mathcal{P}_q . Hence, by the definition of the characteristic function and the

Jensen's Formula, we have

$$\begin{aligned}
I_1 &= \int_{S_m(r)} \log \left\{ \sum_{\mu \in T} \frac{\|f(z)\|^{n+1}}{|C(<f(z), a_{\mu(0)}>, \dots, <f(z), a_{\mu(n)}>)|} \right\} \eta_m(z) \\
&\leq \int_{S_m(r)} \log \|f(z)\|^{n+1} \eta_m(z) - \int_{S_m(r)} \log |C(f_0, \dots, f_n)| \eta_m(z) + O(1) \\
&= (n+1)T_f(r) - N(r, \frac{1}{C(f_0, \dots, f_n)}) + O(1).
\end{aligned}$$

Now denote by $g_{\mu(l)} = \frac{<f(z), a_{\mu(l)}>}{<f(z), a_{\mu(0)}>} = \frac{h_{\mu(l)}}{h_{\mu(0)}}$ for $0 \leq l \leq n$. Then

$$\begin{aligned}
&\frac{|C(h_{\mu(0)}, h_{\mu(1)}, \dots, h_{\mu(n)})|}{|h_{\mu(0)} \bar{h}_{\mu(1)} \cdots \bar{h}_{\mu(n)}^{[n]}|} = \frac{\begin{vmatrix} h_{\mu(0)} & h_{\mu(1)} & \cdots & h_{\mu(n)} \\ \bar{h}_{\mu(0)} & \bar{h}_{\mu(1)} & \cdots & \bar{h}_{\mu(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{h}_{\mu(0)}^{[n]} & \bar{h}_{\mu(1)}^{[n]} & \cdots & \bar{h}_{\mu(n)}^{[n]} \end{vmatrix}}{h_{\mu(0)} \cdot \bar{h}_{\mu(1)} \cdots \bar{h}_{\mu(n)}^{[n]}} \\
&= \frac{\begin{vmatrix} 1 & g_{\mu(1)} & \cdots & g_{\mu(n)} \\ 1 & \bar{g}_{\mu(1)} & \cdots & \bar{g}_{\mu(n)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \bar{g}_{\mu(1)}^{[n]} & \cdots & \bar{g}_{\mu(n)}^{[n]} \end{vmatrix}}{\bar{g}_{\mu(1)} \cdot \bar{g}_{\mu(2)}^{[2]} \cdots \bar{g}_{\mu(n)}^{[n]}} = \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \frac{\bar{g}_{\mu(1)}}{g_{\mu(1)}} & \cdots & \frac{\bar{g}_{\mu(n)}}{g_{\mu(n)}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{\bar{g}_{\mu(1)}^{[n]}}{g_{\mu(n)}} & \cdots & \frac{\bar{g}_{\mu(n)}^{[n]}}{g_{\mu(n)}} \end{vmatrix}}{\frac{\bar{g}_{\mu(1)}}{g_{\mu(1)}} \cdot \frac{\bar{g}_{\mu(2)}^{[2]}}{g_{\mu(2)}} \cdots \frac{\bar{g}_{\mu(n)}^{[n]}}{g_{\mu(n)}}} \\
&= \frac{\sum_{i_1 + \dots + i_n \leq \frac{n(n+1)}{2}} \sum_{l=1}^n \frac{|\bar{g}_{\mu(l)}^{[i_l]}|}{|g_{\mu(l)}|}}{\frac{|\bar{g}_{\mu(1)}|}{|g_{\mu(1)}|} \cdot \frac{|\bar{g}_{\mu(2)}^{[2]}|}{|g_{\mu(2)}|} \cdots \frac{|\bar{g}_{\mu(n)}^{[n]}|}{|g_{\mu(n)}|}}.
\end{aligned}$$

By the Jensen's Formula, we have

$$\begin{aligned}
&\int_{S_m(r)} \log \frac{|h_{\mu(0)} \bar{h}_{\mu(1)} \cdots \bar{h}_{\mu(n)}^{[n]}|}{|h_{\mu(0)} h_{\mu(1)} \cdots h_{\mu(n)}|} \eta_m(z) \\
&\leq \sum_{j=0}^n \int_{S_m(r)} \log |\bar{h}_{\mu(j)}^{[j]}| \eta_m(z) - \sum_{j=0}^n \int_{S_m(r)} \log |h_{\mu(j)}| \eta_m(z) + O(1) \\
&\leq \sum_{j=0}^n N\left(r, \frac{1}{\bar{h}_{\mu(j)}^{[j]}}\right) - \sum_{j=0}^n N\left(r, \frac{1}{h_{\mu(j)}}\right) + O(1).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
I_2 &= \int_{S_m(r)} \sum_{\mu \in T} \log \left\{ \frac{|C(< f(z), a_{\mu(0)} >, \dots, < f(z), a_{\mu(n)} >)|}{\prod_{j=0}^n |< f(z), a_{\mu(j)} >|} \right\} \eta_m(z) \\
&\leq \sum_{\mu \in T} \int_{S_m(r)} \log \left\{ \frac{|C(h_{\mu(0)}, h_{\mu(1)}, \dots, h_{\mu(n)})|}{|h_{\mu(0)} \bar{h}_{\mu(1)} \cdots \bar{h}_{\mu(n)}^{[n]}|} \right\} \eta_m(z) \\
&\quad + \sum_{\mu \in T} \int_{S_m(r)} \log \frac{|h_{\mu(0)} \bar{h}_{\mu(1)} \cdots \bar{h}_{\mu(n)}^{[n]}|}{|h_{\mu(0)} h_{\mu(1)} \cdots h_{\mu(n)}|} \eta_m(z) \\
&\leq \sum_{\mu \in T} \sum_{i_1 + \dots + i_n \leq \frac{n(n+1)}{2}} \sum_{l=1}^n \int_{S_m(r)} \log^+ \frac{|\bar{g}_{\mu(l)}^{[i_l]}|}{|g_{\mu(l)}|} \eta_m(z) + O(1) \\
&\quad + \sum_{\mu \in T} \sum_{l=1}^n \int_{S_m(r)} \log^+ \frac{|g_{\mu(l)}|}{|\bar{g}_{\mu(l)}^{[l]}|} \eta_m(z) + \sum_{\mu \in T} \sum_{j=0}^n \left(N(r, \frac{1}{\bar{h}_{\mu(j)}^{[j]}}) - N(r, \frac{1}{h_{\mu(j)}}) \right).
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
&\int_{S_m(r)} \max_K \sum_{k \in K} \lambda_{H_k}(f(z)) \eta_m(z) \\
&\leq \int_{S_m(r)} \max_{\mu \in T} \sum_{j=0}^n \log \frac{\|f(z)\| \|a_{\mu(j)}\|}{|< f(z), a_{\mu(j)} >|} \eta_m(z) \\
&\leq (n+1)T_f(r) - N(r, \frac{1}{C(f_0, \dots, f_n)}) + O(1) \\
&\quad + \sum_{\mu \in T} \sum_{i_1 + \dots + i_n \leq \frac{n(n+1)}{2}} \sum_{l=1}^n \int_{S_m(r)} \log^+ \frac{|\bar{g}_{\mu(l)}^{[i_l]}|}{|g_{\mu(l)}|} \eta_m(z) \\
&\quad + \sum_{\mu \in T} \sum_{l=1}^n \int_{S_m(r)} \log^+ \frac{|g_{\mu(l)}|}{|\bar{g}_{\mu(l)}^{[l]}|} \eta_m(z) + \sum_{\mu \in T} \sum_{j=0}^n \left(N(r, \frac{1}{\bar{h}_{\mu(j)}^{[j]}}) - N(r, \frac{1}{h_{\mu(j)}}) \right).
\end{aligned}$$

Hence, the proof is completed. \square

For the special case when $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m$ and $\zeta(f) = 0$, we get the following corollary, which will play an important role in proving the second main theorem with hypersurfaces in the next section.

Corollary 4.5. *Let $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m$, and let $f = [f_0, \dots, f_n] : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping with zero order and linearly nondegenerate over the field \mathcal{P}_q^0 . Let H_1, \dots, H_p (defining coefficient vectors a_1, \dots, a_p respectively) be hyperplanes located in general position in $\mathbb{P}^n(\mathbb{C})$. Let T be the set of all injective maps $\mu : \{0, 1, \dots, n\} \rightarrow \{1, \dots, p\}$ such that $a_{\mu(0)}, \dots, a_{\mu(n)}$ are linearly independent.*

Then we have

$$\begin{aligned}
& \int_{S_m(r)} \max_K \sum_{k \in K} \lambda_{H_k}(f(z)) \eta_m(z) \\
& \leq \int_{S_m(r)} \max_{\mu \in T} \sum_{j=0}^n \log \frac{\|f(z)\| \|a_{\mu(j)}\|}{|\langle f(z), a_{\mu(j)} \rangle|} \eta_m(z) \\
& \leq (n+1)T_f(r) - N(r, \frac{1}{C(f_0, \dots, f_n)}) + o(T_f(r))
\end{aligned}$$

for all $r = \|z\|$ on a set of logarithmic density one, where the maximum is taken over all subsets K of $\{1, \dots, p\}$ such that a_j ($j \in K$) are linearly independent.

Proof. Denote by $g_{\mu(l)} = \frac{\langle f(z), a_{\mu(l)} \rangle}{\langle f(z), a_{\mu(0)} \rangle}$ for $0 \leq l \leq n$. Then by the definition of the characteristic function, we have

$$T_{g_{\mu(l)}}(r) \leq T_f(r).$$

Note that the order of f is zero, and thus the order of the meromorphic functions $g_{\mu(l)}$ ($0 \leq l \leq n$) on \mathbb{C}^m are all zero. Whenever $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$, by Theorem 3.2 we get that

$$\int_{S_m(r)} \log^+ \left| \frac{\bar{g}_{\mu(l)}^{[i_l]}}{g_{\mu(l)}} \right| \eta_m(z) + \int_{S_m(r)} \log^+ \left| \frac{g_{\mu(l)}}{\bar{g}_{\mu(l)}^{[i_l]}} \right| \eta_m(z) = o(\max_{0 \leq l \leq n} \{T_{g_{\mu(l)}}(r)\}) = o(T_f(r))$$

for all $r = \|z\|$ on a set of logarithmic density one.

Furthermore, by the Jensen's Formula and the definition of characteristic function, we get that for any $\mu \in T$ and $j \in \{0, 1, \dots, n\}$,

$$\begin{aligned}
N\left(r, \frac{1}{h_{\mu(j)}}\right) &= \int_{S_m(r)} \log |h_{\mu(j)}| \eta_m(z) + O(1) \\
&\leq \int_{S_m(r)} \log \max\{|f_0(z)|, \dots, |f_n(z)|\} \eta_m(z) + O(1) \\
&= T_f(r) + o(T_f(r)),
\end{aligned}$$

and thus

$$\lambda_j := \limsup_{r \rightarrow \infty} \frac{\log^+ N(r, \frac{1}{h_{\mu(j)}})}{\log r} \leq \zeta(f) = 0.$$

Hence by Lemma 7.4 (see below in Section 7), we get that for all $\mu \in T$ and $j \in \{0, 1, \dots, n\}$,

$$\begin{aligned}
N\left(r, \frac{1}{\bar{h}_{\mu(j)}^{[j]}}\right) &\leq (1 + o(1)) N\left(r, \frac{1}{h_{\mu(j)}}\right) \\
&\leq N\left(r, \frac{1}{h_{\mu(j)}}\right) + o(T_f(r)).
\end{aligned}$$

Therefore, the theorem immediately follows from Theorem 4.4. \square

By a careful analysis of the case where the hyperplanes are in general position and the map f is linearly nondegenerate, we have the q -difference analogue of the Cartan's second main theorem with hyperplanes from Corollary 4.5.

Theorem 4.6. *Let $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$, and let $f = [f_0, \dots, f_n] : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic map with zero order and linearly nondegenerate over the field \mathcal{P}_q^0 . Let H_j ($1 \leq j \leq p$) (defining polynomials P_1, \dots, P_p of degree one, respectively) be hyperplanes located in general position in $\mathbb{P}^n(\mathbb{C})$. Then*

$$(p - n - 1)T_f(r) \leq \sum_{j=1}^p N\left(r, \frac{1}{P_j \circ f(z)}\right) - N\left(r, \frac{1}{C(f_0, \dots, f_n)}\right) + o(T_f(r))$$

for all $r = \|z\|$ on a set of logarithmic density one, where $C(f_0, \dots, f_n)$ is the q -Casorati determinant of $f = (f_0, \dots, f_n)$.

Proof. Let a_j be the coefficient vectors of the hyperplane H_j , $1 \leq j \leq p$. By the assumption that H_1, \dots, H_p are located in general position in $\mathbb{P}^n(\mathbb{C})$, we can solve the linear equations system

$$\begin{aligned} \langle f, a_{\mu(0)} \rangle &= a_{\mu(0),0}f_0 + \dots + a_{\mu(0),n}f_n, \\ \langle f, a_{\mu(1)} \rangle &= a_{\mu(1),0}f_0 + \dots + a_{\mu(1),n}f_n, \\ &\vdots \\ \langle f, a_{\mu(n)} \rangle &= a_{\mu(n),0}f_0 + \dots + a_{\mu(n),n}f_n, \end{aligned}$$

and get that

$$f_j = \tilde{a}_{\mu(j),0} \langle f, a_{\mu(0)} \rangle + \dots + \tilde{a}_{\mu(j),n} \langle f, a_{\mu(n)} \rangle, \quad 0 \leq j \leq n,$$

where $(\tilde{a}_{\mu(i),j})_{(n+1) \times (n+1)}$ is the inverse matrix of $(a_{\mu(i),j})_{(n+1) \times (n+1)}$. Hence for any $\mu \in T \subset \{1, \dots, p\}$, $\#T = n + 1$, there exists a positive number A such that

$$\|f(z)\| \leq A \max_{0 \leq j \leq n} \left\{ \frac{|\langle f(z), a_{\mu(j)} \rangle|}{\|a_{\mu(j)}\|} \right\}.$$

It is clear that for any given $z \in \mathbb{C}^m$, there always exists a $\mu \in T \subset \{1, \dots, p\}$, $\#T = n + 1$, such that

$$0 \leq \frac{|\langle f(z), a_{\mu(0)} \rangle|}{\|a_{\mu(0)}\|} \leq \dots \leq \frac{|\langle f(z), a_{\mu(n)} \rangle|}{\|a_{\mu(n)}\|} \leq \frac{|\langle f(z), a_j \rangle|}{\|a_j\|},$$

for $j \neq \mu(i)$, $i = 0, 1, \dots, n$. Hence we obtain

$$\prod_{j=1}^p \frac{\|f(z)\| \|a_j\|}{|\langle f(z), a_j \rangle|} \leq A^{p-n-1} \max_{\mu \in T} \prod_{i=0}^n \left\{ \frac{\|f(z)\| \|a_{\mu(i)}\|}{|\langle f(z), a_{\mu(i)} \rangle|} \right\}.$$

Combining this with Corollary 4.5 gives

$$\begin{aligned} \sum_{j=1}^p m_f(r, H_j) &= \int_{S_m(r)} \log \prod_{j=1}^p \frac{\|f(z)\| \|a_j\|}{|\langle f(z), a_j \rangle|} \eta_m(z) \\ &\leq \int_{S_m(r)} \max_{\mu \in T} \sum_{j=0}^n \log \frac{\|f(z)\| \|a_{\mu(j)}\|}{|\langle f(z), a_{\mu(j)} \rangle|} \eta_m(z) + O(1) \\ &\leq (n+1)T_f(r) - N\left(r, \frac{1}{C(f_0, \dots, f_n)}\right) + o(T_f(r)) \end{aligned}$$

for all $r = \|z\|$ on a set of logarithmic density one. And then by the first main theorem, the theorem is immediately obtained. \square

Remark 4.7. Set $L = \frac{\prod_{j=1}^p H_j(f)}{C(f)}$. Clearly, Both $\prod_{j=1}^p H_j(f)$ and $C(f)$ are entire functions on \mathbb{C}^m . By the Jensen's Formula, we have

$$\begin{aligned}
& N\left(r, \frac{1}{L}\right) - N(r, L) \\
&= \int_{S_m(r)} \log |L(z)| \eta_m(z) + O(1) \\
&= \sum_{j=1}^p \int_{S_m(r)} \log |H_j(f)(z)| \eta_m(z) - \int_{S_m(r)} \log |C(f)(z)| \eta_m(z) + O(1) \\
&= \sum_{j=1}^p N\left(r, \frac{1}{H_j(f)}\right) - N\left(r, \frac{1}{C(f)}\right) + O(1).
\end{aligned}$$

Hence the conclusion of Theorem 4.5 can be written as

$$(p - (n + 1))T_f(r) \leq N(r, \frac{1}{L}) - N(r, L) + o(T_f(r)),$$

which is a q -difference counterpart of the Gundersen-Hayman version of the Cartan's the second main theorem with hyperplanes in several complex variables [17].

5. q -DIFFERENCE ANALOGUE OF THE SECOND MAIN THEOREM FOR HYPERSURFACES

We recall lemmas on Corvaja and Zannier's filtration [10]. Details for proofs can be found in [10, 34, 1].

For a fixed big integer α , denote by V_α the space of homogeneous polynomials of degree α in $\mathbb{C}[x_0, \dots, x_n]$.

Lemma 5.1. [10, 34, 1] *Let $\gamma_1, \dots, \gamma_n$ be homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_n]$ and assume that they define a subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension zero. Then for all large α ,*

$$\dim \frac{V_\alpha}{(\gamma_1, \dots, \gamma_n) \cap V_\alpha} = \deg \gamma_1 \cdots \deg \gamma_n.$$

Throughout of this paper, we shall use the lexicographic ordering on n -tuples $(i_1, \dots, i_n) \in \mathbb{N}^n$ of natural numbers. Namely, $(j_1, \dots, j_n) > (i_1, \dots, i_n)$ if and only if for some $b \in \{1, \dots, n\}$ we have $j_l = i_l$ for $l < b$ and $j_b > i_b$. Given an n -tuple $(\mathbf{i}) = (i_1, \dots, i_n)$ of non-negative integers, we denote $\sigma(\mathbf{i}) := \sum_j i_j$.

Let $\gamma_1, \dots, \gamma_n \in \mathbb{C}[x_0, \dots, x_n]$ be the homogeneous polynomials of degree d that define a zero-dimensional subvariety of $\mathbb{P}^n(\mathbb{C})$. We now recall Corvaja and Zannier's filtration of V_α . Arrange, by the lexicographic order, the n -tuples $(\mathbf{i}) = (i_1, \dots, i_n)$ of non-negative integers such that $\sigma(\mathbf{i}) \leq \frac{\alpha}{d}$. Define the spaces $W_{(\mathbf{i})} = W_{\alpha, (\mathbf{i})}$ by

$$W_{(\mathbf{i})} = \sum_{(\mathbf{e}) \geq (\mathbf{i})} \gamma_1^{e_1} \cdots \gamma_n^{e_n} V_{\alpha - d\sigma(\mathbf{e})}.$$

Clearly, $W_{(0, \dots, 0)} = V_\alpha$ and $W_{(\mathbf{i})} \supset W_{(\mathbf{i}')} if $(\mathbf{i}') > (\mathbf{i})$, so the $W_{(\mathbf{i})}$ is a filtration of V_α .$

Next lemma is a result about the quotients of consecutive spaces in the filtration.

Lemma 5.2. [10, 34, 1] *There is an isomorphism*

$$\frac{W_{(\mathbf{i})}}{W_{(\mathbf{i}')}} \cong \frac{V_{\alpha-d\sigma(\mathbf{i})}}{(\gamma_1, \gamma_n) \cap V_{\alpha-d\sigma(\mathbf{i})}}.$$

Furthermore, we may choose a basis of $\frac{W_{(\mathbf{i})}}{W_{(\mathbf{i}')}} from the set containing all equivalence classes of the form: $\gamma_1^{i_1} \cdots \gamma_n^{i_n} \rho$ modulo $W_{(\mathbf{i}')}$ with ρ being a monomial in x_0, \dots, x_n with total degree $\alpha - d\sigma(\mathbf{i})$.$

Combining Lemma 5.1 and Lemma 5.2, we have the following result.

Lemma 5.3. [10, 34, 1] *There exists an integer α_0 dependent only on $\gamma_1, \dots, \gamma_n$ such that*

$$\Delta_{(\mathbf{i})} := \dim \frac{W_{(\mathbf{i})}}{W_{(\mathbf{i}')}} = d^n$$

provided $d\sigma(\mathbf{i}) < \alpha - \alpha_0$. Also, for the remaining n -tuples (\mathbf{i}) , $\dim \frac{W_{(\mathbf{i})}}{W_{(\mathbf{i}')}} is bounded (by $\dim V_{\alpha_0}$).$

Next, we extend Lemma 4.3 to the case of algebraic independence over \mathcal{P}_q^0 .

Lemma 5.4. *Let $q \in \mathbb{C}^m \setminus \{0\}$. For a positive integer M , set $\mathcal{J}_\alpha = \{(i_0, \dots, i_n) \in \mathbb{N}_0^{n+1} : i_0 + \dots + i_n = \alpha\}$, $I_j \in \mathcal{J}_\alpha$ for all $j \in \{1, \dots, M\}$. Then the meromorphic map $f = [f_0, \dots, f_n] : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ with zero order satisfies $\tilde{C}(f) = C(f^{I_1}, \dots, f^{I_M}) \not\equiv 0$ if and only if the entire functions f_0, \dots, f_n are algebraically nondegenerate over the field \mathcal{P}_q^0 .*

Proof. Set $g_j = f^{I_j}$ ($j = 1, \dots, M$). Then $g = [g_1 : \dots : g_M]$ is a meromorphic map from \mathbb{C}^m into $\mathbb{P}^{M-1}(\mathbb{C})$. According to the definition of the Nevanlinna-Cartan's characteristic function,

$$\begin{aligned} T_g(r) &= \int_{S_m(r)} \log \max_{1 \leq j \leq M} \{|g_j(z)|\} \eta_m(z) + O(1) \\ &= \int_{S_m(r)} \log \max_{1 \leq j \leq M} \{|f^{I_j}(z)|\} \eta_m(z) + O(1) \\ &\leq \int_{S_m(r)} \log \max_{0 \leq j \leq n} \{|f_j(z)|^\alpha\} \eta_m(z) + O(1) \\ &= \alpha T_f(r) + O(1). \end{aligned}$$

Hence, by the definition of the order, we have $\varsigma(g) \leq \varsigma(f) = 0$. Note that f is algebraically nondegenerate (over the field \mathcal{P}_q^0) if and only if g is linearly nondegenerate (over the field \mathcal{P}_q^0). Then by Lemma 4.3 we get that g is linearly nondegenerate over the field \mathcal{P}_q^0 if and only if $C(g) = C(g_1, \dots, g_M) \not\equiv 0$, and thus we complete the proof of the lemma. \square

Now we give the second main theorem for hypersurfaces in general position, in which if the hypersurfaces reduce to hyperplanes and the map reduces to a linearly nondegenerated one, then by taking $\alpha = d = d_j = 1$ the result, stated as follows, implies Theorem 4.6.

Theorem 5.5. *Let $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$, and let $f = [f_0, \dots, f_n] : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic map with zero order and algebraically nondegenerate over the field \mathcal{P}_q^0 . Let Q_j ($1 \leq j \leq p$) (defining homogeneous polynomials D_j of degree d_j)*

be hypersurfaces of degree d_j ($1 \leq j \leq p$) located in general position in $\mathbb{P}^n(\mathbb{C})$. Let d be the least common multiple of the d_j . Then there exists a large positive integer α which is divisible by d , such that

$$\begin{aligned} & (p - n - 1)T_f(r) \\ & \leq \sum_{j=1}^p \frac{1}{d_j} N\left(r, \frac{1}{D_j \circ f(z)}\right) - \frac{1}{\frac{\alpha^{n+1}}{(n+1)!} + O(\alpha^n)} N\left(r, \frac{1}{C(f^{I_1}, \dots, f^{I_M})}\right) + o(T_f(r)) \end{aligned}$$

for all $r = \|z\|$ on a set of logarithmic density one, where $I_j = (i_{j0}, \dots, i_{jn})$, $\#I_j = i_{j0} + \dots + i_{jn} = \alpha$, and $M = \binom{\alpha + n}{n}$.

Proof. Let $f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ with reduced representation $f = [f_0, \dots, f_n]$. Let Q_1, \dots, Q_p be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ located in general position, D_j be the homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_n]$ of degree d_j defining Q_j .

Firstly, we assume that Q_j have the same degree d . For any given $z \in \mathbb{C}^m$, there exists a renumbering $\{i_1, \dots, i_p\}$ of the indices $\{1, \dots, p\}$ such that

$$|D_{i_1} \circ f(z)| \leq |D_{i_2} \circ f(z)| \leq \dots \leq |D_{i_p} \circ f(z)|.$$

Since Q_1, \dots, Q_p are in general position, by Hilbert's Nullstellensatz [37], for any integer k , $0 \leq k \leq n$, there is an integer $m_k \geq d$ such that

$$x_k^{m_k} = \sum_{j=1}^{n+1} b_{kj}(x_0, \dots, x_n) D_{i_j}(x_0, \dots, x_n),$$

where b_{kj} , $1 \leq j \leq n+1$, $0 \leq k \leq n$, are the homogeneous forms with coefficients in \mathbb{C} of degree $m_k - d$. So

$$|f_k(z)|^{m_k} \leq c_1 \|f(z)\|^{m_k - d} \max\{|D_{i_1}(f)(z)|, \dots, |D_{i_{n+1}}(f)(z)|\}.$$

Then we have

$$\prod_{j=1}^p \frac{\|f(z)\|^d}{|D_{i_j} \circ f(z)|} \leq c_1^{p-n} \prod_{k=1}^n \frac{\|f(z)\|^d}{|D_{i_k} \circ f(z)|}.$$

Hence we get that

$$(4) \quad \sum_{j=1}^p m_f(r, Q_j) \leq \int_{S_n(r)} \max_{\{i_1, \dots, i_n\}} \left\{ \log \prod_{k=1}^n \frac{\|f(z)\|^d}{|D_{i_k} \circ f(z)|} \right\} \eta_m(z) + O(1).$$

Pick n distinct polynomials $\gamma_1, \dots, \gamma_n \in \{D_1, \dots, D_p\}$. By the assumption that Q_j 's are in general position, these polynomials define a subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension zero. For a fixed integer α , which will be chosen later, denote by V_α the space of homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_n]$ of degree α . In the above, we recall a filtration $W_{(\mathbf{i})}$ of V_α with

$$\Delta_{(\mathbf{i})} = \dim \frac{W_{(\mathbf{i})}}{W_{(\mathbf{i}')}} = d^n$$

for any $(\mathbf{i}') > (\mathbf{i})$ consecutive n -tuples with $\sigma(\mathbf{i}) \leq \frac{\alpha}{d} - n$, where $\alpha_0 = nd$.

Set $M = M_\alpha := \dim V_\alpha$. We now recall Covaja and Zannier's choice of a suitable basis $\{\psi_1, \psi_2, \dots, \psi_M\}$ for V_α in the following way. We start with the last nonzero $W_{(\mathbf{i})}$ and pick any basis of it. Then we continue inductively as follows: suppose $(\mathbf{i}') > (\mathbf{i})$ are consecutive n -tuples such that $d\sigma(\mathbf{i}), d\sigma(\mathbf{i}') \leq \alpha$ and assume that we

have chosen a basis of $W_{(i')}$. It follows directly from the definition that we may pick representatives in $W_{(i)}$ for the quotient space $\frac{W_{(i)}}{W_{(i')}}$, of the form $\gamma_1^{i_1}, \dots, \gamma_n^{i_n} \rho$, where $\rho \in V_{\alpha-d\sigma(i)}$. We extend the previously constructed basis in $W_{(i')}$ by adding these representatives. In particular, we have obtained a basis for $W_{(i)}$ and our inductive procedure may go on unless $W_{(i)} = V_\alpha$, in which case we stop. In this way, we obtain a basis $\{\psi_1, \dots, \psi_M\}$ for V_α .

We now estimate $\log \prod_{t=1}^M \psi_t \circ f(z)$. Let ψ be an element of the basis, constructed with respect to $\frac{W_{(i)}}{W_{(i')}}$, so we may write $\psi = \gamma_1^{i_1} \dots \gamma_n^{i_n} \rho$, where $\rho \in V_{\alpha-d\sigma(i)}$. Then we have

$$\begin{aligned} |\psi \circ f(z)| &\leq |\gamma_1 \circ f(z)|^{i_1} \dots |\gamma_n \circ f(z)|^{i_n} |\rho \circ f(z)| \\ &\leq c_2 |\gamma_1 \circ f(z)|^{i_1} \dots |\gamma_n \circ f(z)|^{i_n} \|f(z)\|^{\alpha-d\sigma(i)}, \end{aligned}$$

where c_2 is a positive constant depending only on ψ , but not on f or z . Observe that there are precisely $\Delta_{(i)}$ such functions ψ in our basis. Hence,

$$\begin{aligned} \log |\psi_t \circ f(z)| &\leq i_1 \log |\gamma_1 \circ f(z)| + \dots + i_n \log |\gamma_n \circ f(z)| + (\alpha - d\sigma(i)) \log \|f(z)\| + c_3 \\ &= i_1 (\log |\gamma_1 \circ f(z)| - \log \|f(z)\|^d) + \dots + i_n (\log |\gamma_n \circ f(z)| - \log \|f(z)\|^d) \\ &\quad + \alpha \log \|f(z)\| + c_3 \\ &= - \sum_{j=1}^n i_j \log \frac{\|f(z)\|^d}{|\gamma_j \circ f(z)|} + \alpha \log \|f(z)\| + c_3, \end{aligned}$$

where c_3 depends only on the ψ 's, but not on f or z . Now taking the sum, we get

$$\begin{aligned} \log \prod_{t=1}^M |\psi_t \circ f(z)| &= \sum_{t=1}^M \log |\psi_t \circ f(z)| \\ &\leq - \sum_{t=1}^M \sum_{j=1}^n i_j \log \frac{\|f(z)\|^d}{|\gamma_j \circ f(z)|} + M\alpha \log \|f(z)\| + Mc_3 \\ &\leq - \left(\sum_{(i)} \Delta_{(i)} i_j \right) \sum_{j=1}^n \log \frac{\|f(z)\|^d}{|\gamma_j \circ f(z)|} + M\alpha \log \|f(z)\| + Mc_3 \\ &\leq - \Delta \log \prod_{j=1}^n \frac{\|f(z)\|^d}{|\gamma_j \circ f(z)|} + M\alpha \log \|f(z)\| + Mc_3, \end{aligned}$$

where the summations are taken over the n -tuples with $\sigma(i) \leq \frac{\alpha}{d}$, and $\Delta := \sum_{(i)} \Delta_{(i)} i_j$.

Now let ϕ_1, \dots, ϕ_M be a fixed basis of V_α . Then $\{\psi_1, \dots, \psi_M\}$ can be written as linear forms L_1, \dots, L_M in ϕ_1, \dots, ϕ_M such that $\psi_t(f) = L_t(F)$, where

$$F = (\phi_1(f) : \dots : \phi_M(f)) : \mathbb{C}^m \rightarrow \mathbb{P}^{M-1}(\mathbb{C}).$$

The linear forms L_1, \dots, L_M are linearly independent, and so the assumption that f is algebraically nondegenerate over the field \mathcal{P}_q^0 implies that F is linearly nondegenerate over the field \mathcal{P}_q^0 . By the definition of Nevanlinna-Cartan's function and the first main theorem, we have

$$T_F(r) \leq \alpha T_f(r) + O(1),$$

and thus the order of F is not greater than the order of f , namely they are both zero. Therefore,

$$\begin{aligned}
& \log \prod_{j=1}^n \frac{\|f(z)\|^d}{|\gamma_j \circ f(z)|} \\
& \leq -\frac{1}{\Delta} \log \prod_{t=1}^M |\psi_t \circ f(z)| + \frac{M\alpha}{\Delta} \log \|f(z)\| + \frac{Mc_3}{\Delta} \\
& = \frac{1}{\Delta} \log \prod_{t=1}^M \frac{\|F(z)\|}{|L_t \circ F(z)|} - \frac{M}{\Delta} \log \|F(z)\| + \frac{M\alpha}{\Delta} \log \|f(z)\| + \frac{Mc_3}{\Delta}.
\end{aligned}$$

Since there are only finitely many choices $\{\gamma_1, \dots, \gamma_n\} \subset \{D_1, \dots, D_p\}$, we have a finite collection of linear forms L_1, \dots, L_u . Then (4), together with the above inequality, yields

$$\begin{aligned}
& \sum_{j=1}^p m_f(r, Q_j) \\
& \leq \int_{S_m(r)} \max_{\{i_1, \dots, i_n\}} \log \prod_{k=1}^n \frac{\|f(z)\|^d}{|D_{i_k} \circ f(z)|} \eta_m(z) + O(1) \\
& \leq \frac{1}{\Delta} \int_{S_m(r)} \max_K \log \prod_{j \in K} \frac{\|F(z)\| \|L_j\|}{|L_j \circ F(z)|} \eta_m(z) - \frac{M}{\Delta} T_F(r) + \frac{M\alpha}{\Delta} T_f(r) + c_4,
\end{aligned}$$

where \max_K is taken over all subsets K of $\{1, \dots, u\}$ such that linear forms $\{L_j\}_{j \in K}$ are linearly independent, and c_4 is constant independent of r . Now applying Corollary 4.5 to the meromorphic map $F : \mathbb{C}^m \rightarrow \mathbb{P}^{M-1}(\mathbb{C})$ and the hyperplanes defined by the linear forms L_1, \dots, L_u , which are located in general position in $\mathbb{P}^{M-1}(\mathbb{C})$, we get that

$$\begin{aligned}
& \sum_{j=1}^p m_f(r, Q_j) \\
& \leq \frac{1}{\Delta} \int_{S_m(r)} \max_K \log \prod_{j \in K} \frac{\|F(z)\| \|L_j\|}{|L_j \circ F(z)|} \eta_m(z) - \frac{M}{\Delta} T_F(r) + \frac{M\alpha}{\Delta} T_f(r) + c_4 \\
& \leq \frac{1}{\Delta} \left\{ MT_F(r) - N(r, \frac{1}{C(F_1, \dots, F_M)}) + o(T_f(r)) \right\} - \frac{M}{\Delta} T_F(r) + \frac{M\alpha}{\Delta} T_f(r) + c_4 \\
& \leq \frac{M\alpha}{\Delta} T_f(r) - \frac{1}{\Delta} N(r, \frac{1}{C(F_1, \dots, F_M)}) + o(T_f(r))
\end{aligned}$$

for all $r = \|z\|$ on a set of logarithmic density one, where $F_j = \phi_j \circ f$ ($1 \leq j \leq M$).

Since $\phi_j \in V_\alpha$, we may assume that for all $j \in \{1, \dots, M\}$, $F_j = f^{I_j} = f_0^{i_{j0}} \dots f_n^{i_{jn}}$ where $I_j = (i_{j0}, \dots, i_{jn})$ and $\sharp I_j = i_{j0} + \dots + i_{jn} = \alpha$. Then from Lemma 5.4 it follows that

$$C(F_1, \dots, F_M) = C(f^{I_1}, \dots, f^{I_M}) \neq 0.$$

Now we will estimate the coefficient term on $\frac{M\alpha}{\Delta}$ and $\frac{1}{\Delta}$ by modifying the reasoning in [34, pp. 222–223]. Firstly, we have

$$M = \binom{\alpha + n}{n} = \frac{(\alpha + n)!}{\alpha! n!} = \frac{\alpha^n}{n!} + O(\alpha^{n-1}).$$

Secondly, since the number of nonnegative integer k -tuples with the sum $\leq S$ is equal to the number of nonnegative integer $(k+1)$ -tuples with the sum exactly $S \in \mathbb{Z}$, which is $\binom{S+k}{k}$, and since the sum below is independent of j , we have that, for α divisible by d , and for every j ,

$$\begin{aligned} \sum_{(i)} i_j &= \frac{1}{n+1} \sum_{(i)} \sum_{j=1}^{n+1} i_j = \frac{1}{n+1} \sum_{(i)} \frac{\alpha}{d} \\ &= \frac{1}{n+1} \binom{\frac{\alpha}{d} + n}{n} \frac{N}{d} = \frac{\alpha^{n+1}}{d^{n+1}(n+1)!} + O(\alpha^n), \end{aligned}$$

where the sum $\sum_{(i)}$ is taken over the nonnegative integer $(n+1)$ -tuples with sum exactly $\frac{\alpha}{d}$. Combining this and Lemma 5.3, we have, for every $1 \leq j \leq n$,

$$\Delta = \sum_{(i)} i_j \Delta_{(i)} = \frac{\alpha^{n+1}}{d(n+1)!} + O(\alpha^n),$$

where again the summations are taken over the n -tuples with the sum $\leq \frac{\alpha}{d}$. Hence, we have

$$\frac{M\alpha}{\Delta} = \frac{d(n+1)\alpha^n + O(\alpha^{n-1})}{\alpha^n + O(\alpha^{n-1})} = \frac{d(n+1) + O(\alpha^{-1})}{1 + O(\alpha^{-1})} = d(n+1) + o(1),$$

and

$$\frac{1}{\Delta} = \frac{1}{\frac{\alpha^{n+1}}{d(n+1)!} + O(\alpha^n)}.$$

By the first main theorem, $m_f(r, Q_j) = dT_f(r) - N(r, \frac{1}{D_j \circ f(z)}) + O(1)$. Therefore, we get that

$$\begin{aligned} & d(p-n-1)T_f(r) \\ & \leq \sum_{j=1}^p N(r, \frac{1}{D_j \circ f(z)}) - \frac{1}{\frac{\alpha^{n+1}}{d(n+1)!} + O(\alpha^n)} N(r, \frac{1}{C(f^{I_0}, \dots, f^{I_M})}) + o(T_f(r)) \end{aligned}$$

for all $r = \|z\|$ on a set of logarithmic density one.

Now we assume that D_1, \dots, D_p are the homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_n]$ of degree d_j defining hypersurfaces Q_j , $1 \leq j \leq p$, and d is the least common multiple of the d_j . Then all $D_j^{\frac{d}{d_j}}$ ($1 \leq j \leq p$) are of degree d , and thus by the above mentioned part we have

$$\begin{aligned} & (p-n-1)T_f(r) \\ & \leq \sum_{j=1}^p \frac{1}{d} N\left(r, \frac{1}{D_j^{\frac{d}{d_j}} \circ f(z)}\right) - \frac{1}{\frac{\alpha^{n+1}}{(n+1)!} + O(d\alpha^n)} N\left(r, \frac{1}{C(f^{I_1}, \dots, f^{I_M})}\right) + o(T_f(r)) \end{aligned}$$

for all $r = \|z\|$ on a set of logarithmic density one. Note that if $z_0 \in \mathbb{C}^m$ is a zero of $D_j \circ f$ with multiplicity k , then z_0 is zero of $D_j^{\frac{d}{d_j}} \circ f$ with multiplicity $k\frac{d}{d_j}$. This

implies that

$$\frac{1}{d}N\left(r, \frac{1}{D_j^{\frac{d}{d_j}} \circ f(z)}\right) \leq \frac{1}{d_j}N\left(r, \frac{1}{D_j \circ f(z)}\right).$$

Therefore, we obtain

$$\begin{aligned} & (p - n - 1)T_f(r) \\ & \leq \sum_{j=1}^p \frac{1}{d_j}N\left(r, \frac{1}{D_j \circ f(z)}\right) - \frac{1}{\frac{\alpha^{n+1}}{(n+1)!} + O(d\alpha^n)}N\left(r, \frac{1}{C(f^{I_1}, \dots, f^{I_M})}\right) + o(T_f(r)) \end{aligned}$$

for all $r = \|z\|$ on a set of logarithmic density one. Note that $O(d\alpha^n)$ can be simply written as $O(\alpha^n)$ for a large number α , and thus we complete the proof of this theorem. \square

6. DIFFERENCE ANALOGUES OF GENERALIZED PICARD-TYPE THEOREMS

Fujimoto [13] and Green [16] gave a natural generalization of the Picard's theorem by showing that if $f : \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ omits $n + p$ hyperplanes in general position where $p \in \{1, \dots, n + 1\}$, then the image of f is contained in a linear subspace of dimension at most $\lfloor \frac{n}{p} \rfloor$. In 2014, Halburd, Korhonen and Tohge [19] proposed a q -difference analogue of the general Picard-type theorem for homomorphic curves with zero order.

Theorem 6.1. [19, Theorem 6.1] *Let $f : \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve with zero order. Let $q \in \mathbb{C} \setminus \{0\}$, and let $p \in \{1, \dots, n + 1\}$. If $p + n$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ have forward invariant preimages under f with respect to the rescaling $\tau(z) = qz$, then the image of f is contained in a projective linear subspace over \mathcal{P}_q^0 of dimension $\leq \lfloor \frac{n}{p} \rfloor$.*

Here we say that the pre-image of $H(f(z))$ for a meromorphic mapping $f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ intersecting a hyperplane H of $\mathbb{P}^n(\mathbb{C})$ is forward invariant with respect to the rescaling $\tau = qz$ if $\tau(H(f)^{-1}) \subset H(f)^{-1}$ where $\tau(H(f)^{-1})$ and $H(f)^{-1}$ are considered to be multi-sets in which each point is repeated according to its multiplicity. By this definition the (empty and thus forward invariant) pre-images of the usual Picard exceptional values become special cases of forward invariant pre-images.

In this section we extend Theorem 6.1 to the case of meromorphic mappings $f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ intersecting hyperplanes in general position which have forward invariant preimages under f with respect to the rescaling $\tau(z) = qz$, $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$.

Theorem 6.2. *Let $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$, let $p \in \{1, \dots, n + 1\}$. Assume that f is a meromorphic mapping from \mathbb{C}^m into $\mathbb{P}^n(\mathbb{C})$ of zero order. If $p + n$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ have forward invariant preimages under f with respect to the rescaling $\tau(z) = qz$, then the image of f is contained in a projective linear subspace over \mathcal{P}_q^0 of dimension $\leq \lfloor \frac{n}{p} \rfloor$.*

Before proving Theorem 6.2, we need two lemmas as follows.

Lemma 6.3. *Let $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$, and $f = [f_0, \dots, f_n]$ be a meromorphic mapping from \mathbb{C}^m into $\mathbb{P}^n(\mathbb{C})$ of zero order, and let all zeros of f_0, \dots, f_n be forward*

invariant with respect to the rescaling $\tau(z) = qz$. If $\frac{f_i}{f_j} \notin \mathcal{P}_q^0$ for all $i, j \in \{0, \dots, n\}$ such that $i \neq j$, then f is linearly nondegenerate over \mathcal{P}_q^0 .

Proof. Assume that the conclusion is not true, that is there exist $A_0, \dots, A_n \in \mathcal{P}_q^0$ such that

$$(5) \quad A_0 f_0 + \dots + A_{n-1} f_{n-1} = A_n f_n$$

and such that not all A_j are identically zero. Without loss of generality we may assume that none of A_j are identically zero. Since all zeros of f_0, \dots, f_n are forward invariant with respect to the rescaling $\tau(z) = qz$ and since $A_0, \dots, A_n \in \mathcal{P}_q^0$, we can choose a meromorphic function F on \mathbb{C}^m such that $FA_0 f_0, \dots, FA_n f_n$ are holomorphic functions on \mathbb{C}^m without common zeros and such that the preimages of all zeros of $FA_0 f_0, \dots, FA_n f_n$ are forward invariant with respect to the rescaling $\tau(z) = qz$. Then we have

$$(6) \quad \limsup_{r \rightarrow \infty} \frac{\log^+ \left(N(r, \frac{1}{F}) + N(r, F) \right)}{\log r} = 0$$

and $FA_0 f_0, \dots, FA_{n-1} f_{n-1}$ cannot have any common zeros.

Denote $g_j := FA_j f_j$ for $0 \leq j \leq n$. Then $T_G(r)$ is well defined for $G = [g_0, \dots, g_{n-1}]$, which is a holomorphic mapping from \mathbb{C}^m into $\mathbb{P}^{n-1}(\mathbb{C})$. Then by the definition of Nevanlinna-Cartan's characteristic function and the Jensen's Formula, we have

$$\begin{aligned} T_{G(r)} &= \int_{S_m(r)} \log \|G\| \eta_m(z) + O(1) \\ &\leq \int_{S_m(r)} \log |F(z)| \eta_m(z) + \int_{S_m(r)} \log \|f(z)\| \eta_m(z) \\ &\quad + \sum_{j=0}^{n-1} \int_{S_m(r)} \log^+ |A_j| \eta_m(z) + O(1) \\ &\leq N\left(r, \frac{1}{F}\right) - N(r, F) + T_f(r) + \sum_{j=0}^{n-1} T_{A_j}(r) \end{aligned}$$

which together with (6) imply that the order of G satisfies $\zeta(G) = 0$.

Assume that the meromorphic mapping $G : \mathbb{C}^m \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$ is linearly nondegenerate over \mathcal{P}_q^0 . Then by Lemma 4.3, it follows that $C(g_0, \dots, g_{n-1}) \neq 0$. Define the following hyperplanes

$$H_j : w_j = 0, \quad 0 \leq j \leq n-1,$$

and

$$H_n : w_0 + w_1 + \dots + w_{n-1} = 0,$$

where $[w_0, \dots, w_{n-1}]$ is a homogeneous coordinate system of $\mathbb{P}^{n-1}(\mathbb{C})$. So, we have $H_j(G(z)) = g_j(z)$ for $0 \leq j \leq n-1$ and

$$H_n(G(z)) = g_0(z) + \dots + g_{n-1}(z) = F(z)A_n(z)f_n(z) = g_n(z).$$

Clearly, the $n+1$ hyperplanes H_0, \dots, H_n are located in general position of $\mathbb{P}^{n-1}(\mathbb{C})$. Hence by Theorem 4.6 we have

$$\begin{aligned} T_G(r) &= ((n+1) - (n-1) - 1) T_G(r) \\ &\leq \sum_{j=0}^n N\left(r, \frac{1}{g_j}\right) - N\left(r, \frac{1}{C(g_0, \dots, g_{n-1})}\right) + o(T_G(r)) \end{aligned}$$

for all r on a set of logarithmic density one.

Since the preimages of all zeros of g_0, \dots, g_n are forward invariant with respect to $\tau(z) = qz$, all zeros of $g_j, j = 0, \dots, n-1$, are zeros of the q -Casorati determinant $C(g_0, \dots, g_{n-1})$ with the same or higher multiplicity. Moreover, since g_0, \dots, g_n do not have any common zeros, it follows in particular that for each $z_0 \in \mathbb{C}^m$ such that $g_n(z_0) = 0$ with multiplicity m_0 there exists $k_0 \in \{0, \dots, n-1\}$ such that $g_{k_0} := FA_{k_0}f_{k_0} \neq 0$. Using (5) we may write

$$C(g_0, \dots, g_{n-1}) = C(g_0, \dots, g_{k_0-1}, g_n, g_{k_0+1}, \dots, g_{n-1})$$

which implies that $C(g_0, \dots, g_{n-1})$ has a zero at z_0 with multiplicity m_0 at least. Also, at any common zero the functions $g_{j_k} := FA_{j_k}f_{j_k}$ with multiplicities m_{j_k} , $k = 1, \dots, l$, where $\{j_1, \dots, j_l\} \subset \{1, \dots, n\}$ and $l \leq n-2$, the Casorati determinant $C(g_0, \dots, g_{n-1})$ has a zero of multiplicity $\geq \sum_{k=1}^l m_{j_k}$. This implies

$$\sum_{j=0}^n N\left(r, \frac{1}{g_j}\right) \leq N\left(r, \frac{1}{C(g_0, \dots, g_{n-1})}\right).$$

Hence, it follows that $T_G(r) = o(T_G(r))$ for all r on a set of logarithmic density one, which is a contradiction.

Therefore, the meromorphic mapping $G : \mathbb{C}^m \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$ is linearly degenerate over \mathcal{P}_q^0 , and thus there exist $B_0, \dots, B_{n-1} \in \mathcal{P}_q^0$ such that

$$B_0 f_0 + \dots + B_{n-2} f_{n-2} = B_{n-1} f_{n-1}$$

and such that not all B_j are identically zero. By repeating similar discussions as above it follows that there exist $L_i, L_j \in \mathcal{P}_q^0$ such that

$$L_i f_i = L_j f_j$$

for some $i \neq j$ and not all L_i and L_j are identically zero. This contradicts the condition that $\frac{f_i}{f_j} \notin \mathcal{P}_q^0$ for all $\{i, j\} \subset \{0, \dots, n\}$. Therefore, the proof is completed. \square

The following lemma is a q -difference analogue of the Borel's theorem.

Lemma 6.4. *Let $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$, and $f = [f_0, \dots, f_n]$ be a meromorphic mapping from \mathbb{C}^m into $\mathbb{P}^n(\mathbb{C})$ of zero order, and let all zeros of f_0, \dots, f_n be forward invariant with respect to the rescaling $\tau(z) = qz$. Let*

$$S_1 \cup \dots \cup S_l$$

be the partition of $\{0, 1, \dots, n\}$ formed in such a way that i and j are in the same class S_k if and only if $\frac{f_i}{f_j} \in \mathcal{P}_q^0$. If

$$f_0 + \dots + f_n = 0,$$

then

$$\sum_{j \in S_k} f_j = 0$$

for all $k \in \{1, \dots, l\}$.

Proof. Suppose that $i \in S_k$, $k \in \{0, \dots, l\}$. Then by the condition of the lemma, $f_i = A_{i,j_k} f_{j_k}$ for some $A_{i,j_k} \in \mathcal{P}_q^0$ whenever the indexes i and j_k are in the same class S_k . This implies that

$$0 = \sum_{k=0}^n f_k = \sum_{k=1}^l \sum_{i \in S_k} A_{i,j_k} f_{j_k} = \sum_{k=1}^l B_k f_{j_k}$$

where $B_k = \sum_{i \in S_k} A_{i,j_k} \in \mathcal{P}_q^0$. This gives that f_{j_1}, \dots, f_{j_l} are linearly degenerate over \mathcal{P}_q^0 . Hence by Lemma 6.3 all B_k ($k = 1, \dots, l$) are identically zero. Thus it follows

$$\sum_{i \in S_k} f_i = \sum_{i \in S_k} A_{i,j_k} f_{j_k} = B_k f_{j_k} \equiv 0$$

for all $k = \{1, \dots, l\}$. □

Proof of Theorem 6.2. We modify the method of proof of [16, Theorem 1] as follows. Denote $T = \{1, \dots, n+p\}$. Let H_j be defined by

$$H_j : \quad h_{j0}(z)w_0 + \dots + h_{jn}(z)w_n = 0 \quad (j \in T)$$

where $[w_0, \dots, w_n]$ is a homogeneous coordinate system of $\mathbb{P}^n(\mathbb{C})$. Since $\{H_j\}_{j \in T}$ are in general position of $\mathbb{P}^n(\mathbb{C})$, any $n+2$ of H_j satisfy a linear relation with nonzero coefficients in \mathbb{C} . By conditions of the theorem, holomorphic functions

$$g_j := H_j(f) = h_{j0}f_0 + \dots + h_{jn}f_n$$

satisfy

$$\{\tau(g_j^{-1}(\{0\}))\} \subset \{g_j^{-1}(\{0\})\}$$

for all $j \in T$, where $\{\cdot\}$ denotes a multiset with counting multiplicities of its elements. We say that $i \sim j$ if $g_i = \beta g_j$ for some $\beta \in \mathcal{P}_q^0 \setminus \{0\}$. Hence

$$T = \bigcup_{j=1}^l S_j$$

for some $l \in T$.

Firstly, assume that the complement of S_k has at least $n+1$ elements for some $k \in \{1, \dots, l\}$. Choose an element $s_0 \in S_k$, and denote $U = (T \setminus S_k) \cup \{s_0\}$. Then U contains at least $n+2$ elements, and thus there is a subset $U_0 \subset U$ such that $U_0 \cap S_k = \{s_0\}$ and $\#U_0 = n+2$. Therefore there exists $\beta_j \in \mathbb{C} \setminus \{0\}$ such that

$$\sum_{j \in U_0} \beta_j H_j = 0.$$

Hence,

$$\sum_{j \in U_0} \beta_j g_j = \sum_{j \in U_0} \beta_j H_j(f) = 0.$$

Without loss of generality, we may assume that $U_0 = \{s_1, \dots, s_{n+1}\} \cup \{s_0\}$. It is easy to see from above discussion that all of zeros of $\beta_j g_j$ ($j \in U_0$) are forward invariant with respect to the rescaling $\tau(z) = qz$, and

$$G := [\beta_{s_0} g_{s_0} : \beta_{s_1} g_{s_1} : \dots : \beta_{s_{n+1}} g_{s_{n+1}}]$$

is a meromorphic mapping from \mathbb{C}^m into $\mathbb{P}^{n+1}(\mathbb{C})$ with zero order. Furthermore, $\frac{\beta_i g_i}{\beta_{s_0} g_{s_0}} \notin \mathcal{P}_q^0$ for any $i \in U_0 \setminus \{s_0\}$, thus $i \not\sim s_0$. Hence by Lemma 6.4 we have

$\beta_{s_0} g_{s_0} = 0$, and thus $H_{s_0}(f(z)) \equiv 0$. This means that the image $f(\mathbb{C}^m)$ is included in the hyperplane H_{s_0} of $\mathbb{P}^n(\mathbb{C})$.

Secondly, assume that the set $T \setminus S_k$ has at most n elements. Then S_k has at least p elements for all $k = 1, \dots, l$. This implies that

$$l \leq \frac{n+p}{p}.$$

Let V be any subset of T with $\#V = n+1$. Then $\{H_j\}_{j \in V}$ are linearly independent. Denote $V_k := V \cap S_k$. Then we have

$$V = \bigcup_{k=1}^l V_k.$$

Since each set V_k gives raise to $\#(V_k - 1)$ equations over the field \mathcal{P}_q^0 , it follows that there are at least

$$\begin{aligned} \sum_{k=1}^l (\#V_k - 1) &= n+1-l \geq n+1 - \frac{n+p}{p} \\ &= n - \left(\frac{n}{p}\right) \end{aligned}$$

linear independent relations over the field \mathcal{P}_q^0 . This means that the image of f is contained in a linear subspace over \mathcal{P}_q^0 of dimension $\leq [\frac{n}{p}]$. The proof of the theorem is finished. \square

According to the definition of forward invariant pre-image, the following result is an extension of the Picard's theorem under the "zero order" growth condition.

Theorem 6.5. *Let f be a meromorphic mapping with zero order from \mathbb{C}^m into $\mathbb{P}^n(\mathbb{C})$, and let $\tau(z) = qz$, where $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$. If $\tau((f, H_j)^{-1}) \subset (f, H_j)^{-1}$ (counting multiplicity) hold for p distinct hyperplanes $\{H_j\}_{j=1}^p$ in general position in $\mathbb{P}^n(\mathbb{C})$, and if $p > n+1$, then $f(z) \equiv f(qz)$.*

Proof. By Theorem 6.2, the image of f is contained in a projective linear subspace over \mathcal{P}_q^0 of dimension $\leq [\frac{n}{p}]$. By the assumption $p > n$ it follows $[\frac{n}{p}] = 0$. Hence $f(z) = f(qz)$. The proof of Theorem 6.5 is thus completed. \square

The following corollary follows immediately from the above theorem, which can be seen as a q -difference counterpart of the well-known five-value theorem for meromorphic functions in the complex plane due to Nevanlinna.

Corollary 6.6. *Let f be a meromorphic function with zero order on \mathbb{C}^m and let $\tau(z) = qz$, where $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$. If $\tau((f(a_j))^{-1}) \subset (f(a_j))^{-1}$ (counting multiplicity) hold for p distinct numbers $\{a_j\}_{j=1}^p$ in \mathbb{P} , and if $p > 2$, then $f(z) \equiv f(qz)$.*

7. q -DIFFERENCE ANALOGUES OF THE TUMURA-CLUNIE THEOREM IN SEVERAL COMPLEX VARIABLES

The Clunie lemma [9] for meromorphic functions of one variable in Nevanlinna theory has been a powerful tool in the field of complex differential equations and related fields, particularly the lemma has been used to investigate the value distribution of certain differential polynomials; see [9] for the original versions of these

results, as well as [20, 26]. A slightly more general version of the Clunie lemma can be found in [21, pp. 218–220]; see also [26, Lemma 2.4.5]. In 2007, the additional assumptions in the He-Xiao version of the Clunie lemma have been removed by Yang and Ye in [41, Theorem 1]. A generalized Clunie lemma for meromorphic functions of several complex variables was proved in [29]; for some special cases refer to see [22, 23]. Recently, Hu and Yang [24] extended the classical Tumura-Clunie theorem ([20, Theorem 3.9] and [30]) for meromorphic functions of one variable to that of meromorphic functions of several complex variables.

In [2], Barnett, Halburd, Korhonen and Morgan proved a basic q -difference Clunie lemma for meromorphic functions of one variable. In [27], Laine and Yang gave a generalized q -difference Clunie lemma in one complex variable by adapting the method of Yang and Ye [41] to the q -difference case. Recently, Wang [38] attempted to extend the generalized q -difference Clunie lemma due to Laine and Yang to the case for meromorphic functions in several complex variables, and he apply the extended result to complex partial q -difference equations. Unfortunately, Wang's proof relies on a version of lemma on q -difference quotients that, as we mention above, has a gap in the proof. Here, we restate the Wang's conclusion of generalized q -difference Clunie lemma in several complex variables based on our version (Theorem 3.2) of the q -difference quotient lemma in several complex variables. Since the proof is almost the same as the proof of [38, Theorem 2.1], we omit it. Note that other main results in [38], such as Theorems 3.1–3.4, should also be revised similarly.

Define complex partial q -difference polynomials as follows

$$(7) \quad P(z, w) = \sum_{\lambda \in I} a_{\lambda}(z) w(z)^{l_{\lambda_0}} w(q_{\lambda_1} z)^{l_{\lambda_1}} \cdots w(q_{\lambda_i} z)^{l_{\lambda_i}},$$

$$(8) \quad Q(z, w) = \sum_{\mu \in J} b_{\mu}(z) w(z)^{l_{\mu_0}} w(q_{\mu_1} z)^{l_{\mu_1}} \cdots w(q_{\mu_j} z)^{l_{\mu_j}},$$

$$(9) \quad U(z, w) = \sum_{\nu \in K} c_{\nu}(z) w(z)^{l_{\nu_0}} w(q_{\nu_1} z)^{l_{\nu_1}} \cdots w(q_{\nu_k} z)^{l_{\nu_k}},$$

where all coefficients $a_{\lambda}(z)$, $b_{\mu}(z)$ and $c_{\nu}(z)$ are small functions with respect to the function $w(z)$ meromorphic on \mathbb{C}^m , I, J, K are three finite sets of multi-indices, and $q_s \in \mathbb{C}^m \setminus \{0\}$, ($s \in \{\lambda_1, \dots, \lambda_i, \mu_1, \dots, \mu_j, \nu_1, \dots, \nu_k\}$).

Theorem 7.1. *Let w be a nonconstant meromorphic function of zero order on \mathbb{C}^m , and let $P(z, w)$, $Q(z, w)$, and $U(z, w)$ be complex partial q -difference polynomials as (7), (8) and (9) satisfying a complex partial q -difference equation of the form*

$$(10) \quad U(z, w)P(z, w) = Q(z, w).$$

Assume that the total degree of $U(z, w)$ is equal to n , and the total degree of $Q(z, w)$ is less than or equal to n , and that $U(z, w)$ contains just one term of maximal total degree in $w(z)$ and its shifts. If $q_s = (\tilde{q}_s, \dots, \tilde{q}_s) \in \mathbb{C}^m \setminus \{0\}$ for all $s \in \{\lambda_1, \dots, \lambda_i, \mu_1, \dots, \mu_j, \nu_1, \dots, \nu_k\}$, then we have

$$m(r, P(z, w)) = o(T_w(r))$$

for all r on a set of logarithmic density one.

Next we prove a q -difference counterpart of the Hu-Yang's version of Tumura-Clunie theorem in several complex variables [24] as follows. Take a q -difference polynomial of several complex variables

$$(11) \quad G(z, f) = \sum_{\lambda \in J} b_\lambda(z) \sum_{j=1}^{\tau_\lambda} f(q_{\lambda,j}z)^{\mu_{\lambda,j}},$$

where $\max_{\lambda \in J} \sum_{j=1}^{\tau_\lambda} \mu_{\lambda,j} = n$, and $q_{\lambda,j} \neq 1$ for at least one of the constants $q_{\lambda,j}$. Moreover, we assume that the coefficients in (11) are meromorphic functions on \mathbb{C}^m and small with respect to the function f , which is meromorphic on \mathbb{C}^m .

Theorem 7.2. *Let f be a meromorphic function of zero order on \mathbb{C}^m such that*

$$(12) \quad N\left(r, \frac{1}{f}\right) + N(r, f) = o(T_f(r)),$$

and $q_{\lambda,j} = (\tilde{q}_{\lambda,j}, \dots, \tilde{q}_{\lambda,j}) \in \mathbb{C}^m \setminus \{0\}$. Then the difference polynomial (11) of $f(z)$ and its shifts, of maximal total degree n , satisfies

$$N\left(r, \frac{1}{G}\right) \neq o(T_f(r)).$$

For the proof of Theorem 7.2, we first need the Tumura-Clunie theorem of several complex variables due to Hu and Yang.

Lemma 7.3. [24, Theorem 2.1] *Suppose that f is meromorphic and not constant in \mathbb{C}^m , that*

$$g = f^n + P_{n-1}(f),$$

where $P_{n-1}(f)$ is a differential polynomial of degree at most $n-1$ in f , and that

$$N(r, f) + N\left(r, \frac{1}{g}\right) = o(T_f(r)).$$

Then

$$g = \left(f + \frac{\alpha}{n}\right)^n,$$

where α is a meromorphic function in \mathbb{C}^m , small with respect to f , and determined by the terms of degree $n-1$ in $P_{n-1}(f)$ and by g .

The following result is an extension of the relations of counting functions of $N(r, f(qz))$ and $N(r, f(z))$ to the case of several complex variables. We omit the proof, since it is almost the same as the proof of [44, Theorem 1.3] in the case of one variable due to Zhang and Korhonen.

Lemma 7.4. *Let f be a nonconstant meromorphic function on \mathbb{C}^m with zero order, and $q \in \mathbb{C}^m \setminus \{0\}$. Then*

$$N(r, f(qz)) = (1 + o(1))N(r, f(z))$$

on a set of lower logarithmic density one.

We note here that by Theorem 3.2 and Lemma 7.4, we can restate the conclusion [39, Theorem 9] (an extension of [44, Theorem 1.1]) that

$$T_{f(qz)}(r) = (1 + o(1))T_{f(z)}(r)$$

holds on a set of lower logarithmic density one, where the nonconstant meromorphic function f on \mathbb{C}^m is of zero order and $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$.

Proof of Theorem 7.2. Suppose that the conclusion is not true, and we assume that

$$N\left(r, \frac{1}{G}\right) = o(T_f(r)).$$

To prove this theorem, we propose to follow the idea in the proof of [28, Theorem 1]. Since the difference polynomial (11) of $f(z)$ and its shifts is of maximal total degree n ,

$$\begin{aligned} G(z, f) &= \sum_{\lambda \in J} b_\lambda(z) \sum_{j=1}^{\tau_\lambda} f(q_{\lambda,j}z)^{\mu_{\lambda,j}} \\ &= \sum_{\lambda \in J} b_\lambda(z) \sum_{j=1}^{\tau_\lambda} \left[\left(\frac{f(q_{\lambda,j}z)}{f(z)} \right)^{\mu_{\lambda,j}} \cdot f(z)^{\mu_{\lambda,j}} \right] \\ &:= \sum_{j=0}^n \tilde{b}_j(z) f(z)^j, \end{aligned}$$

where each of the coefficients $\tilde{b}_j(z)$ ($j = 1, \dots, n$) is the sum of finitely many terms of type

$$b_\lambda(z) \left(\frac{f(q_{\lambda,j}z)}{f(z)} \right)^{\mu_{\lambda,j}}.$$

Hence

$$\frac{G(z, f)}{\tilde{b}_n(z)} = f(z)^n + \sum_{j=0}^{n-1} \frac{\tilde{b}_j(z)}{\tilde{b}_n(z)} f(z)^j.$$

Note that $q_{\lambda,j} = (\tilde{q}_{\lambda,j}, \dots, \tilde{q}_{\lambda,j}) \in \mathbb{C}^m \setminus \{0\}$ and all the coefficient functions $b_\lambda(z)$ ($\lambda \in J$) are small with respect to f . Then by Theorem 3.2 we get that for all $j = 1, \dots, n$,

$$m(r, \tilde{b}_j) = o(T_f(r))$$

holds for all r on a set of logarithmic density one. Moreover, by the assumption (12) and Lemma 7.4 we have

$$N(r, \tilde{b}_j) = o(T_f(r)),$$

and thus

$$T(r, \tilde{b}_j) = o(T_f(r)), \quad j \in \{0, 1, \dots, n\}$$

and

$$N\left(r, \frac{1}{\frac{G(z, f)}{\tilde{b}_n(z)}}\right) = o(T_f(r)).$$

for all r on a set of logarithmic density one. Hence by Lemma 7.3 we may write

$$\frac{G(z, f)}{\tilde{b}_n(z)} = \left(f(z) + \frac{\alpha(z)}{n} \right)^n,$$

where $T_\alpha(r) = o(T_f(r))$. This implies that

$$(13) \quad N\left(r, \frac{1}{f(z) + \frac{\alpha(z)}{n}}\right) = o(T_f(r)).$$

Together with (12) and (13), it follows from the second main theorem for meromorphic functions with small function targets on \mathbb{C}^m (it is mentioned in [8, Theorem

2.1] that the conclusion is easily extended from the second main theorem for small function targets in one variable due to Yamanoi [42] by the standard process of averaging over the complex lines in \mathbb{C}^m) that

$$T_f(r) \leq N\left(r, \frac{1}{f}\right) + N(r, f) + N\left(r, \frac{1}{f(z) + \frac{\alpha(z)}{n}}\right) + o(T_f(r)) = o(T_f(r))$$

for all r on a set of logarithmic density one. Hence we get a contradiction. \square

Open Question. It is open whether our main theorems (essentially, Theorem 3.2) are still true for a general constant $q \in \mathbb{C}^m \setminus \{0\}$ instead of the $q = (\tilde{q}, \dots, \tilde{q}) \in \mathbb{C}^m \setminus \{0\}$.

Acknowledgement. The first author would like to thank Professor Pei-Chu Hu (Shandong University) for informing on the recent progress on Tumura-Clunie type theorems in several complex variables and sending the pdf file of the reference [24].

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(Tingbin Cao) DEPARTMENT OF MATHEMATICS, NANCHANG UNIVERSITY, JIANGXI 330031, P. R. CHINA

E-mail address: `tbcao@ncu.edu.cn`

(Risto Korhonen) DEPARTMENT OF PHYSICS AND MATHEMATICS, UNIVERSITY OF EASTERN FINLAND, P. O. BOX 111, FI-80101 JOENSUU, FINLAND

E-mail address: `risto.korhonen@uef.fi`